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EFFECTIVE SOLUTION OF THE FUNDAMENTAL QUASI-PERIODIC PROBLEMS OF THE THEORY OF ELASTICITY FOR A PLANE WITH CUTS DISTRIBUTED ALONG A STRAIGHT LINE†

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The method of the Riemann boundary-value problem for a denumerable set of contours is used to solve the fundamental quasi-periodic problems of the theory of elasticity for a plane with cuts distributed along a straight line. The solutions are obtained in explicit form as "ordinary" "corrected" Cauchy-type integrals along a denumerable set of segments of the real axis, and uniformly converging series of simple fractions whose coefficients are found from an infinite system of linear algebraic equations. In a number of cases the solutions of the system are found explicitly, for example, when the boundary conditions of the problem are periodic or decrease near infinity as some power function of degree less than minus one. The system has a unique solution in all cases.

Formulas are obtained for the stress intensity factors and their asymptotic expressions for the cuts situated near infinity. Numerical examples are given for the quasi-linear problem of the theory of cracks.

These problems were first formulated in [1, 2] and studied using the method of discrete Fourier transforms. The preference for the method used in the present paper is due to the fact that it does not contain such additional transformations as the direct and inverse Fourier transformations.

THE PERIODIC and certain generalized periodic problems studied by many authors using different approaches are special cases of the quasi-periodic problems dealt with in this paper. A detailed survey of the literature dealing with these problems is given in [3–5].

1. FORMULATION OF THE PROBLEM

Let a homogeneous, isotropic elastic plane $z = x + iy$ be cut along the line L consisting of the segment $L_k = [kT - a, kT + a]$ ($k = 0, \pm 1, \ldots; a < T/2$) of the real x axis and on the edges $L^{\pm}$ of the cut L either normal and shear stresses $(\sigma_y, \tau_{xy})^{\pm}$ (the first problem) or partial derivatives with respect to x of the displacement components $(u', v')^{\pm}$ (the second problem) are specified, or stresses are specified on L^+, while derivatives of the displacement components (the mixed problem) are given on L^-. We shall regard the specified functions as H-continuous and uniformly bounded on L, i.e. the values of these functions will not exceed, in modulo, the same positive constant. In the general case the boundary conditions are non-periodic, therefore the stress–strain state realized in this case will also be non-periodic.

In the present case we have the following formulas [6] for the stresses σ_x, σ_y, τ_{xy}, the rotation ω and derivatives with respect to x of the displacement components u', v' in a plane with a cut along the line L:

$$\sigma_x+\sigma_y=4\operatorname{Re}\Phi(z),\quad 2\mu\omega=(1+\varkappa)\operatorname{Im}\Phi(z)$$
$$\sigma_y-i\tau_{xy}=\Phi(z)+\Omega(\bar z)+(z-\bar z)\overline{\Phi'(z)} \tag{1.1}$$

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where μ, $\varkappa$ are the elastic constants of the material. The functions $\Phi(z)$, $\Omega(z)$ are holomorphic in the plane with a cut along the line L, and at the ends of the cut they may become infinite of order less than unity. Since the point ∞ is a singular point for these functions, it follows that we must also specify how these functions behave near ∞.

We shall consider the stress–strain state determined by the functions $\Phi(z)$, $\Omega(z)$ which increase in modulus, as $z\to\infty$, outside any fixed, sufficiently smooth ε-neighbourhood $U_\varepsilon(L)$ of the line L, not more rapidly than the expression $M|z|^\lambda$, $M>0$, $\lambda<1$.

2. THE FIRST PROBLEM

Boundary-value problems

The functions [6]

$$\Phi_{1,2}(z)=\Phi(z)\pm\Omega(z) \tag{2.1}$$

represent the solutions of the Riemann boundary-value problems

$$\Phi_1^+(t)+\Phi_1^-(t)=2g_1(t),\quad t\in L \tag{2.2}$$

$$\Phi_2^+(t)-\Phi_2^-(t)=2g_2(t),\quad t\in L \tag{2.3}$$

$$2g_{1,2}(t)=(\sigma_y-i\tau_{xy})^+\pm(\sigma_y-i\tau_{xy})^-$$

for a denumerable set of segments L_k, of which the line L consists, in the class of functions which increase, outside $U(L)$ as $z\to\infty$, in modulus, not more rapidly than $M|z|^\lambda$, $\lambda<1$, and may, at the ends of the segments, become infinite of order less than unity, i.e. they belong to the class h_0 [7]. The functions $g_{1,2}(t)$ are H-continuous and uniformly bounded on the line L by definition.

The general solution of problem (2.3) has the form [8]

$$\Phi_2(z)=B+\frac{z}{\pi i}\int_L g_2(t)\frac{dt}{t(t-z)} \tag{2.4}$$

where B is a complex constant and the integral along L converges absolutely and uniformly in z in any bounded domain not containing points of the line L.

The solution of problem (2.2)

Since the distribution of the segments L_k is periodic, it follows that we can use the following function [8] as the canonical function of class h_0 of problem (2.2):

$$X(z)=\left(\sin\frac{\pi(z+a)}{T}\sin\frac{\pi(z-a)}{T}\right)^{-1/2} \tag{2.5}$$

or

$$X_1(z)=X(z)\sin\frac{\pi z}{T} \tag{2.6}$$

where we shall regard $X(z)$ as a branch holomorphic in the plane with a cut along the line L, satisfying the condition $\lim X_1(z)=1$ as $z=iy\to\pm i\infty$. Then $X_1(z)\to 1$ as $z\to\infty$ along the points of any fixed set D_ε consisting of the angles $\varepsilon<|\arg z|<\pi-\varepsilon$, $0<\varepsilon<\pi/2$.

Outside any fixed neighbourhood $U(L)$ the above functions will satisfy the inequalities $0<|X(z)|\leqslant M$, $m\leqslant|X_1(z)|\leqslant M$ where m, M are positive constants. The functions $(z-\bar z)X'(z)$, $(z-\bar z)X_1'(z)$ are also uniformly bounded outside $U(L)$ and tend to zero as $z\to\infty$ along the points of the set D_ε. The function $X_1(z)$ is T-periodic and $X(z)$ is $2T$-periodic.

Since the function $X_1(z)$ becomes zero of the first order at the points kT ($k=0,\pm1,\ldots$), it follows that we can write the particular solution of problem (2.2) in the form

$$F_1(z) = X_1(z)\,R(z), \qquad R(z) = \sum_{k=-\infty}^{\infty} \frac{1}{\pi i\,(z-kT)} \int_{L_k} \frac{g_1(\tau)}{X_1^+(\tau)} \frac{\tau - kT}{\tau - z}\,d\tau \tag{2.7}$$

where the series converges, by virtue of the uniform boundedness of the functions $g_1(\tau)$ and $(\tau - kT)/X_1^+(\tau)$ on L, absolutely and uniformly outside any fixed neighbourhood $U(L)$. Then the function $F_0(z) = \Phi_1(z) = F_1(z)$ will be a solution of the homogeneous problem corresponding to problem (2.2), and by virtue of the properties of the function $X_1(z)$ listed above, the quotient $Q(z) = F_0(z)/X_1(z)$ will represent a meromorphic function with simple poles kT ($k = 0, \pm 1, \ldots$) which increases in modulus outside $U(L)$, as $z \to \infty$, less rapidly than $M|z|^\lambda$, $\lambda < 1$. Therefore [9]

$$Q(z) = A + A_0 z^{-1} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} A_k \left(\frac{1}{z-kT} + \frac{1}{kT}\right) \tag{2.8}$$

where the constants A_k are such, that a series in any bounded domain not containing the points kT, will converge uniformly. Clearly, this will be true if and only if the following series converges:

$$\sum_{k=-\infty}^{\infty} A_k (k^2 + c)^{-1} \tag{2.9}$$

where c is any fixed positive number. Then

$$\Phi_1(z) = F_1(z) + X_1(z) Q(z) \tag{2.10}$$

Requiring that the displacements be single-valued on going around the cuts L_n, we obtain, using relations (1.1), (2.1)–(2.3), (2.8) and (2.10), the following system for determining the constants A_k:

$$\sum_{k=-\infty}^{\infty} \delta_{n-k} A_k = iB_n + \frac{\varkappa - 1}{2(\varkappa+1)} P_n, \qquad n = 0, \pm 1, \ldots \tag{2.11}$$

$$\delta_n = \int_0^b \frac{2x}{x^2 - \pi^2 n^2} \frac{\sin x\,dx}{(\sin^2 b - \sin^2 x)^{1/2}}, \qquad b = \frac{\pi a}{T} \tag{2.12}$$

$$B_n = -\int_{L_n} F_1(t)\,dt, \qquad P_n = -2i \int_{L_n} g_1(t)\,dt$$

where P_n is the principal vector of external forces acting on the edges of the cut L_n, and the function $F_1(t)$ is governed by formula (2.7) in which $X_1(t)$ must be replaced by $X_1^+(t)$. The solution of system (2.11) should be sought in the space Π of the sequences $\{A_k\}$ for which series (2.8), (2.9) converge and $|Q(z)| \leq M|z|^\lambda$, $\lambda < 1$ for large $z \notin U(L)$.

Properties of system (2.11)

Let us write system (2.11) in the form

$$A_n = \sum_{k=-\infty}^{\infty} \alpha_{nk} A_k + C_n, \qquad n = 0, \pm 1, \ldots \tag{2.13}$$

$$\alpha_{nn} = 0, \quad \alpha_{nk} = -\delta_0^{-1}\delta_{n-k}, \quad n \neq k; \quad C_n = \delta_0^{-1}\left(iB_n + \frac{\varkappa-1}{2(\varkappa+1)} P_n\right)$$

According to (2.12) $\delta_0 > 0$, and the remaining coefficients $\delta_{-n} = \delta_n < 0$. Therefore all $\alpha_{nk} = \alpha_{kn} > 0$ and

$$\sum_{k=-\infty}^{\infty} |\alpha_{nk}| = -2\delta_0^{-1} \sum_{m=1}^{\infty} \delta_m = 2\delta_0^{-1} \int_0^b \sum_{m=1}^{\infty} \frac{2x}{\pi^2 m^2 - x^2} \frac{\sin x\,dx}{(\sin^2 b - \sin^2 x)^{1/2}}$$

from which we obtain, after summing the series [10]

$$\sum_{k=-\infty}^{\infty} |\alpha_{nk}| = \sum_{n=-\infty}^{\infty} |\alpha_{nk}| = 1 - \frac{\pi}{\delta_0}; \qquad n, k = 0, \pm 1, \ldots \tag{2.14}$$

Similarly we can show that for any value of $a \in (0, T/2)$ positive numbers c and $\theta < 1$ exist such that

$$\sum_{n=-\infty}^{\infty} |\alpha_{nk}| (n^2 + c)^{-m} < \theta (k^2 + c)^{-m}; \quad k = 0, \pm 1, \ldots; \quad m = 1, 2 \tag{2.15}$$

From (2.14) and (2.15) it follows that the infinite matrix $\|\alpha_{nk}\|$ defines the compression operator in the spaces of bounded sequences l_∞, of absolutely summable sequences l_1, and also in the Banach space Π_m $(m = 1, 2)$ of the sequences $\{A_k\}$ for which the following series converges:

$$\sum_{k=-\infty}^{\infty} |A_k| (k^2 + c)^{-m}$$

From all this it follows [11] that system (2.13), and hence (2.11), are solvable in the above spaces and have unique solutions, which can be found using the method of successive approximations. Moreover, the solutions can also be found in the spaces l_1 and l_∞ using the reduction method [12].

Since in the case in question the sequence $\{C_n\} \in l_\infty$, therefore the inclusions $l_\infty \subset \Pi \subset \Pi_2$ imply that system (2.11) has a unique solution in the space Π which will not be bounded.

If the inequalities $|C_n| \leq M |n|^{-1-\lambda}$, $\lambda > 0$ hold for large n, which happens, for example, when the boundary conditions of the problem as $t \to \infty$ decrease as $o(|t|^{-1-\lambda})$, then [13]

$$A_n = \frac{\delta_0}{2\pi i} \int_{|t|=1} \frac{C(t)}{\delta(t)} \frac{dt}{t^{n+1}}, \quad C(t) = \sum_{n=-\infty}^{\infty} C_n t^n, \quad \delta(t) = \sum_{n=-\infty}^{\infty} \delta_n t^n \tag{2.16}$$

The numbers A_n for large n also satisfy the inequalities $|A_n| \leq M |n|^{-1-\nu}$, $\forall \nu: 0 < \nu < \min\{\lambda; 1\}$.

Behaviour of the solutions for large z

Following the well-known approach [14] we can show that the function $\Phi_2(z)$, for large $z \in U(L)$ satisfies the inequalities

$$|\Phi_2(z)| \leq M \ln|z|, \quad |(z-\bar{z})\Phi_2'(z)| \leq M \ln|z|, \quad M > 0 \tag{2.17}$$

and cases exist in which $\Phi_2(z)$ increases logarithmically as $z \to \infty$, for example [15], if $g_2(t) = \text{const} \neq 0$ on the segments $L_0, L_1, \ldots,$ and $g_2(t) = 0$ on the remaining segments. The functions $Q(z)$, $\Phi_1(z)$ have identical properties. It follows, therefore, that as $z \to \infty$, the stresses and the rotation will, generally speaking, increase logarithmically. Therefore in the general case we must not, as in the classical case [6], take the values of the stresses and the rotation as $z \to \infty$ as the conditions for finding the still undetermined constants A, B.

Instead, we can use as such conditions, for example, the values of the stresses and the rotation at any finite point $z_0 \notin L$. However, if the functions $\Phi j(z)$, $(z - \bar{z})\Phi_j'(z)$ $(j = 1, 2)$ tend to some defined limits along any curves, e.g. along the arcs, as $z \to \infty$, we can use the values of the stresses and the rotation as $z \to \infty$ along these curves as conditions in determining the constants A, B. Then A, B will be obtained from these conditions and from relations (1.1) and (2.1).

3. THE QUASI-PERIODIC PROBLEM OF THE THEORY OF CRACKS

The stress intensity factors (SIF)

According to relations (2.1), (2.4)–(2.8), (2.10) the functions $\Phi(z)$, $\Omega(z)$ have the following form [7] near the tips $a + nT$ $(n = 0, \pm 1, \ldots)$:

$$\Phi(z) \infty \Omega(z) \infty {}^1/_2 \Phi_1(z) \infty (K_1 - iK_2)_n^+ / (2\sqrt{2(z-a-nT)})$$

$$(K_1 - iK_2)_n^+ = \left(\frac{T}{\pi} \operatorname{tg} \frac{\pi a}{T}\right)^{1/2} \{R(nT + a) + Q(nT + a)\} \tag{3.1}$$

which implies that the numbers $(K_1, K_2)_n^+$ are the SIF in the form given in [3]. Multiplying them by $\sqrt{\pi}$ we obtain the form in [16]. Similarly, near the tip $nT - a$ the SIF are

$$(K_1 - iK_2)_n^- = \left(\frac{T}{\pi}\operatorname{tg}\frac{\pi a}{T}\right)^{1/2}\{R(nT-a)+Q(nT-a)\} \qquad (3.2)$$

Since $Q(z)$ can increase logarithmically as $z\to\infty$, it follows that the SIF can also increase as $\ln|n|$ when $n\to\infty$. In the latter case the periodic system of cracks in question will be unstable.

The behaviour of the stresses near the crack tips is determined in terms of the SIF using the well-known representations [3].

The case of decreasing boundary conditions

Let the boundary conditions specified in $t\to\infty$ decrease as some function $|t|^{-\lambda}$, $\lambda>0$. Then the solutions $\{A_k\}$ of system (2.11) will have the same property as $k\to\infty$ and the solutions of problems (2.2) and (2.3) will have the form

$$\Phi_1(z) = F_1(z) + X_1(z)Q(z), \quad Q(z) = A + \sum_{k=-\infty}^{\infty} A_k(z-kT)^{-1} \qquad (3.3)$$

$$\Phi_2(z) = B + \frac{1}{\pi i}\int_L g_2(t)\frac{dt}{t-z} \qquad (3.4)$$

The functions X_1 and F_1 are found using the formulas (2.6) and (2.7), and the numbers A_k are found from system (2.11) or implicitly from (2.16). In this case the functions $\Phi_j(z)$, $(z-\bar z)\Phi_j'(z)$ $(j=1, 2)$ will be uniformly bounded outside any fixed neighbourhood $U(L)$ and the functions $(z-\bar z)\Phi_j'(z)$ will tend to zero as $z\to\infty$ along the rays originating at the origin of coordinates and situated in the upper or lower half-plane, while the functions $\Phi_1(z)$ and $\Phi_2(z)$ will tend, respectively, to A and B. Then from (1.1) and (2.1) we obtain

$$\sigma_y^\infty - i\tau_{xy}^\infty = A, \quad \frac{1}{2}(\sigma_x^\infty+\sigma_y^\infty) + \frac{4\mu i}{\varkappa+1}\omega^\infty = A+B \qquad (3.5)$$

where σ_x^∞, σ_y^∞, τ_{xy}^∞, ω^∞ are the values of the stresses and the rotation as $z\to\infty$ along the given rays, which should be specified.

Since in the present case the functions $R(nY\pm a)$ and $Q(nY\pm a)$ have the limits 0 and A, respectively, as $n\to\infty$, therefore according to relations (3.1) and (3.2) there exists

$$\lim_{n\to\infty}(K_1 - iK_2)_n^\pm = (\sigma_y^\infty - i\tau_{xy}^\infty)\left(\frac{T}{\pi}\operatorname{tg}\frac{\pi a}{T}\right)^{1/2} \qquad (3.6)$$

The stability of the system of cracks depends, in this case, on the value of the limit (3.6) as well as on the values of the SIF near the tips of a certain finite number of cracks [16, 17].

Let us consider in greater detail the case when the forces specified at the crack edges decrease, as $t\to\infty$, as $O(|t|^{-\lambda})$, $\lambda>1$. Then

$$\Phi_2(z) = B - \frac{1}{\pi i z}\int_L g_2(t)\,dt + \frac{1}{\pi i z}\int_L t g_2(t)\frac{dt}{t-z}$$

$$Q(z) = A + \frac{1}{z}\sum_{k=-\infty}^{\infty} A_k + \frac{T}{z}\sum_{k=-\infty}^{\infty}\frac{kA_k}{z-kT}$$

which yields, for large $z\notin U(L)$,

$$\Phi_2(z) = B - \frac{P}{2\pi z} + o(z^{-\nu}), \quad Q(z) = A + \frac{H}{z} + o(z^{-\nu}) \qquad (3.7)$$

where P is the principal vector of external forces applied to the edges of all cracks, and H is the sum of all numbers A_k. In this case $P\neq\infty$. In order to find H we combine all equations of (2.1). This yields

$$\sum_{k=-\infty}^{\infty} A_k S_k = -iI + \frac{\varkappa - 1}{2(\varkappa + 1)} P, \quad S_k = \sum_{n=-\infty}^{\infty} \delta_{n-k}, \quad I = \int_L F_1(t)\,dt$$

From Eqs (2.7) and (2.12) we find $I = 0$, $S_k = \pi$ $(k = 0, \pm 1, \ldots)$, therefore

$$H = (\varkappa - 1)P / [2\pi(\varkappa + 1)]$$

Since, for large $z \notin U(L)$, the function $F_1(z) = O(z^{-\lambda})$, it follows that according to the relations (2.1), (3.3) and (3.7) we have the following asymptotic relations for large $z \notin U(L)$:

$$\begin{gathered} 2\begin{Bmatrix}\Phi(z)\\ \Omega(z)\end{Bmatrix} = \left(A + \frac{H}{z}\right) X_1(z) + \begin{Bmatrix}1\\ -1\end{Bmatrix}\left(B - \frac{P}{2\pi z}\right) + o(z^{-\nu}) \\ 2\Phi'(z) = AX_1'(z) + \frac{H}{z}\left(X_1'(z) - \frac{X_1(z)}{z}\right) + \frac{P}{2\pi z^2} + o(z^{-1-\nu}) \\ \forall \nu:\ 1 < \nu < \min\{\lambda;\ 2\} \end{gathered} \tag{3.8}$$

The function $X_1(z)$ and constants A, B are found from Eqs (2.6) and (3.5). For large z situated along the arc $\arg z = \varphi$, $\varepsilon < |\varphi| < \pi - \varepsilon$, the representations (3.8), by virtue of the equations $X_1(z) \propto 1$, $X_1'(z) = o(z^{-2})$, will be the same as in the case of a plane with a finite number of cuts [6], apart from the function z^{-1}. The nature of the stress distribution for large $z \notin U(L)$ is defined by Eqs (1.1) and (3.8). In the present case, for large n the SIF, according to relations (3.1), (3.2) and (3.7), have the form

$$\begin{gathered} (K_1 - iK_2)_n^{\pm} = \left(\frac{T}{\pi} \operatorname{tg} \frac{\pi a}{T}\right)^{1/2} \left\{\sigma_y^{\infty} - i\tau_{xy}^{\infty} + \frac{(\varkappa - 1)P}{2\pi T(\varkappa + 1)n}\right\} + o(n^{-\nu}) \\ \forall \nu:\ 1 < \nu < \min\{\lambda;\ 2\} \end{gathered} \tag{3.9}$$

The symmetric case

Let the normal and shear external forces $\sigma(t)$ and $\tau(t)$, symmetrical about the coordinate axes, be applied to the edges of the crack L_0, i.e.

$$(\sigma_y - i\tau_{xy})^+(t) = (\sigma_y - i\tau_{xy})^-(t) = (\sigma_y - i\tau_{xy})^{\pm}(-t) = -\sigma(t) + i\tau(t), \quad t \in L_0$$

while the edges of remaining cracks are stress-free and the stresses and the rotation vanish at ∞. Then we have in the systems (2.11) and (2.13) $P_n = 0$, $B_0 = 0$, and

$$\begin{gathered} iB_n = \frac{8Tn}{\pi} \int_0^a y\xi(y) \int_0^a \frac{x}{\xi(x)} \frac{p(x)\,dx \checkmark}{((y + nT)^2 - x^2)((y - nT)^2 - x^2)}, \quad n \neq 0 \\ \xi(x) = \left(\sin^2 \frac{\pi a}{T} - \sin^2 \frac{\pi x}{T}\right)^{-1/2} \sin \frac{\pi x}{T}, \qquad p(x) = \sigma(x) - i\tau(x) \end{gathered} \tag{3.10}$$

from which it follows that $B_{-n} = -B_n$, $C_{-n} = -C_n$ $(n = 1, 2, \ldots)$.

Let us combine Eq. (2.13) when $n = m$ with the equation when $n = -m$. This yields the homogeneous system

$$A_m + A_{-m} = \sum_{k=1}^{\infty} (\alpha_{mk} + \alpha_{-m,k})(A_k + A_{-k}), \qquad m = 1, 2, \ldots$$

which in the space of sequences Π, has only a zero solution $A_k + A_{-k} = 0$, whence $A_{-k} = -A_k$ $(k = 1, 2, \ldots)$. We also have $A_0 = B_0 = C_0 = 0$. Then the solution of the quasi-periodic system will be given by the functions

$$\Phi(z) = \Omega(z) = \Phi_0(z) = {}^1/_2 X_1(z)(R_0(z) + Q_0(z)) \tag{3.11}$$

$$R_0(z) = -\frac{2}{\pi} \int_0^a \frac{t}{\xi(t)} \frac{p(t)\,dt}{z^2 - t^2}, \qquad Q_0(z) = \sum_{k=1}^{\infty} \frac{2kTA_k}{z^2 - k^2T^2} \tag{3.12}$$

The constants A_k are found from the system

$$\sum_{k=1}^{\infty} (\delta_{n-k} - \delta_{n+k}) A_k = iB_n, \qquad n = 1, 2, \ldots \tag{3.13}$$

In the present case we find that according to relations (3.8) and (3.9) the functions $\Phi(z)$, $\Omega(z)$ as $z \to \infty$ decrease outside $U(L)$ as $O(z^{-\nu})$, and the SIF as $n \to \infty$ decrease as $O(n^{-\nu})$ where $\forall \nu: 1 < \nu < 2$. It is clear that the stresses also have the same property as $z \to \infty$.

The case of similar loads

Let the edges of the cracks L_n be acted upon by loads $(\sigma - i\tau)_n = p_n(\sigma(t) - i\tau(t))$ differing from each other only by the constant complex factors p_n which increase, as $n \to \infty$, not more rapidly than n^{λ}, $\lambda < 1$. Then the solution of the quasi-periodic problem will be given by the functions

$$\Phi(z) = \Omega(z) = \sum_{k=-\infty}^{\infty} p_k \Phi_0(z - kT) \tag{3.14}$$

and the SIF will be

$$(K_1 - iK_2)_n^{\pm} = \sum_{k=-\infty}^{\infty} p_k (M_1 - iM_2)_{n-k}^{\pm} \tag{3.15}$$

where $\Phi_0(z)$ is the solution of the particular problem formulated above, described by the formulas (3.10)–(3.13), $(M_1, M_2)_n^{\pm}$ are the SIF of the particular problem and the series (3.14) and (3.15) converge.

Example 1. Let $p(t) = \sigma(t) - i\tau(t) = \sigma - i\tau = \text{const}$ in the particular problem for the symmetric case. Then the solution of this problem will be given by the function $\Phi(z) = \Omega(z) = (\sigma - i\tau)F(z)$ and the SIF are $(K_1)_n^{\pm} = \sigma M_n^{\pm}$, $(K_2)_n^{\pm} = \tau M_n^{\pm}$, where $F(z)$ is the solution and $M_n^{\pm}$ is the SIF of the particular problem in the case where $p(t) = 1$, defined by (3.10)–(3.13) and (3.1), (3.2), (3.12), respectively.

Table 1 gives, for the period $T = \pi$, the values of the factors $M_n^{\pm}$ as a function of the ratio a/π. In columns 3 and 4 we give for comparison, the quantities $N^{\pm} = \sqrt{\operatorname{tg} a}$ which are the SIF of the periodic problem of the theory of cracks and $K^{\pm} = \sqrt{a}$ which is the SIF for the case of a single crack $[-a, a]$ when a constant unit normal load [3] is applied to the edges of the cracks.

Calculations have shown that for the tips $\pm a$, in greatest danger of fracture, we always have $M_0^{\pm} < N^{\pm}$, $M_0^{\pm} > K^{\pm}$, although in practice the second inequality holds only when $(a/\pi) > 0.3$.

Example 2. In the particular problem $p(t) = \sigma(t) - i\tau(t) = (Y - iX)\delta(t)$, where $\delta(t)$ is the Dirac delta functions, let concentrated forces $X + iY$ and $-X - iY$ be applied to the opposite edges of the crack L_0 at the middle points, respectively. Then $\Phi(z) = \Omega(z) = (Y - iX)F(z)$ and $(K_1)_n^{\pm} = YM_n^{\pm}$, $(K_2)_n^{\pm} = XM_n^{\pm}$ where $F(z)$ is the solution and $M_n^{\pm}$ is the SIF of the particular problem in the case when $p(t) = \delta(t)$. In the present case we have, in formulas (3.10) and (3.12),

$$iB_n = \frac{4nT^2}{\pi^2}\left(\sin\frac{\pi a}{T}\right)\int_0^a \frac{x\xi(x)\,dx}{(x^2 - n^2T^2)^2}, \quad R_0(z) = \frac{T}{\pi^2 z^2}\sin\frac{\pi a}{T}$$

TABLE 1

Tip	a			$\pi - a$	$\pi + a$	$2\pi - a$
a/π	$M_0^{\pm}$	$N^{\pm}$	$K^{\pm}$	M_1^{-}	M_1^{+}	M_2^{-}
0.01	0.177	0.177	0.177	0.0	0.0	0.0
0.05	0.396	0.398	0.396	0.001	0.0	0.0
0.10	0.561	0.570	0.561	0.003	0.002	0.001
0.20	0.794	0.852	0.792	0.023	0.016	0.007
0.30	0.988	1.173	0.971	0.087	0.051	0.029
0.40	1.225	1.754	1.121	0.296	0.153	0.116
0.45	1.482	2.513	1.189	0.629	0.314	0.272

TABLE 2

Tip	a			$\pi-a$	$\pi+a$	$2\pi-a$
a/π	$M_0^{\pm}$	$N^{\pm}$	$K^{\pm}$	M_1^-	M_1^+	M_2^-
0.01	1.796	1.796	1.796	0.0	0.0	0.0
0.05	0.803	0.810	0.803	0.002	0.002	0.001
0.10	0.568	0.587	0.568	0.007	0.005	0.002
0.15	0.464	0.500	0.464	0.013	0.010	0.004
0.20	0.404	0.462	0.402	0.023	0.016	0.006
0.25	0.365	0.450	0.359	0.036	0.023	0.011
0.30	0.340	0.462	0.329	0.056	0.033	0.019
0.35	0.327	0.500	0.304	0.086	0.048	0.032
0.37	0.326	0.527	0.295	0.103	0.056	0.039
0.40	0.332	0.587	0.284	0.136	0.073	0.055
0.45	0.382	0.810	0.268	0.242	0.128	0.111

Table 2 shows, for the period $T=\pi$, the values of the factors $M_n^{\pm}$, $N^{\pm}$, $K^{\pm}$ as a function of the ratio a/π where $N^{\pm}=\sqrt{2}/(\pi\sqrt{\sin 2a})$ are the SIF of the periodic problem and $K^{\pm}=1/(\pi\sqrt{a})$ are the SIF for the case of a single crack where normal concentrated forces of unit magnitude and in opposite directions are applied to the opposite edges of the cracks at their middle points [3].

Calculations showed that for the tips $\pm a$ the factors $M_0^{\pm}$ at $0<(a/\pi)<0.37$ decrease from ∞ to 0.326, and when $0.37<(a/\pi)<\frac{1}{2}$ they increase, and we always have $M_0^{\pm}<N^{\pm}$, $M_0^{\pm}>K^{\pm}$, although in practice the first inequality holds only when $(a/\pi)>0.1$ and the second when $(a/\pi)>0.3$.

4. THE SECOND PROBLEM

In this case [6] to find the functions

$$\Phi_{1,2}(z)=\varkappa\Phi(z)\mp\Omega(z) \tag{4.1}$$

we again have the boundary-value problems (2.2), (2.3), where we must take

$$g_{1,2}(t)=\mu(u'+iv')^+\pm\mu(u'+iv')^-$$

The general solution $\Phi(z)$ of problem (2.3) and particular solution $F_1(z)$ of problem (2.2) are given, depending on the properties of the functions $g_{1,2}(t)$, by (2.3), (2.7) and (3.4). The general solution of problem (2.2) depends on the type of additional conditions which must be specified in order to determine the constants included in this solution.

Let us specify the principal vectors P_n $(n=0, \pm1, \ldots)$ of the external forces acting at the edges of the cuts L_n, bounded in totality, as these conditions. Then, taking the general solution of problem (2.2) in the form (2.10), (2.8) we again obtain, in accordance with Eqs (1.1), (4.1), (2.2), (2.3), the system (2.13) for determining the constants A_k, where we must take

$$C_n=\delta_0^{-1}\left(-i\int_{L_n}F_1(t)\,dt-\frac{\varkappa}{\varkappa+1}P_n\right)$$

Therefore, in this case the results of Secs 2 and 3 hold.

Let the differences s_n $(n=0, \pm1, \ldots)$ of the displacements of the points $nT+T-a$ and $nT+a$, bounded in totality, be given. We will take the solution of problem (2.2) in the form

$$\Phi_1(z)=F_1(z)+X_2(z)Q(z),\quad X_2(z)=X(z)\cos(\pi z/T) \tag{4.2}$$

$$Q(z)=A+\sum_{k=-\infty}^{\infty}A_k\left(\frac{1}{z-kT-T/2}+\frac{1}{kT+T/2}\right) \tag{4.3}$$

The function $X(z)$ is given by formula (2.5), and the numbers A_k are such, that series (4.3) converges and determines the function $Q(z)$ satisfying, for large $z \notin U(L)$, the inequality $|Q(z)| \leq M|z|^{\lambda}$, $\lambda < 1$. The function $X_2(z)$ satisfies, outside any fixed neighbourhood $U(L)$, the inequalities $m \leq |X_2(z)| \leq M$, $M > m > 0$ and tends to $-i$ when $y \to +\infty$ and to i when $y \to -\infty$. Having calculated the differences s_n, from (1.1), (4.1)–(4.3), we again obtain system (2.13) for determining the constants A_k, where

$$C_n = \delta_0^{-1}\Big(-2\mu s_n - \int_{nT+a}^{nT+T-a} F_1(t)\,dt\Big)$$

and we must take $b = (\pi/2) - \pi a/T$ in (2.12).

It follows, therefore, that in this case all the results of Secs 2 and 3 are also valid except for the formulas for determining the constants A, B and the formulas for the SIF.

Let the boundary conditions of the problem decrease as $t \to \infty$, when $|t|^{-\lambda}$, $\lambda > 0$ and let the numbers s_k have the same property as $k \to \infty$. Then the function $Q(z)$ in (4.2) will be given by the formula

$$Q(z) = A + \sum_{k=-\infty}^{\infty} A_k\left(z - kT - \frac{T}{2}\right)^{-1} \tag{4.4}$$

and will have the limit A as $y \to \pm\infty$, while the function $\Phi_2(z)$, defined in this case by formula (3.4), will have the limit B. The function $\Phi_1(z)$ tends, as $y \to \pm\infty$, to the numbers $\mp iA$. Then, according to relations (1.1), (4.1), the expression $2\varkappa(\sigma_y - i\tau_{xy}) : (\varkappa + 1)$ will tend, as $y \to \pm\infty$, to the numbers $B \mp iA$. The expression

$$\frac{\varkappa}{2}(\sigma_x + \sigma_y) + \frac{4\mu\varkappa i}{\varkappa + 1}\omega$$

will also tend to the same numbers as $y \to \pm\infty$.

Let G_1 and G_2 be the values of one of the above expressions as $y \to +\infty$ and $y \to -\infty$, respectively. Then

$$A = (i/2)(G_1 - G_2), \quad B = (1/2)(G_1 + G_2)$$

Formulas (3.1), (3.2), (3.6), (3.9) and (3.16) hold for the SIF. Here the function $Q(z)$ must be determined from (4.3) or (4.4) and then multiplied by $-\mathrm{ctg}(\pi a/T)$.

Notes. 1. The results of Secs 2–4 also remain valid in the case when the boundary conditions of the problems on each segment L_n belong to the class H_0 and increase in modulus as t increases not more rapidly than some function of the form $|t|^{\lambda}$, $\lambda < 1$. In this case the stresses increase as z increases, generally speaking, as $|z|^{\nu}$, $\lambda < \nu < 1$. The SIF exhibit the same type of growth near the tips of the segments L_n as $n \to \infty$.

2. In the case of periodic boundary conditions the solutions of quasi-periodic problems constructed above are identical with the known solutions of the corresponding periodic problems [3, 5, 18–21], i.e. in this case the stress state bounded at ∞ will necessarily be periodic. We note that in the periodic problems discussed in the literature the condition that the stresses are periodic in the elastic domain is specified as the initial condition and is not derived from the periodic boundary conditions only.

3. The validity of (2.16) follows from the results of [22] also in the case when the sequence $\{C_n\} \in l_p$: $\forall p$: $1 \leq p < +\infty$. This occurs, for example, when the boundary conditions of the problem decrease near infinity as a power function of any negative exponent. This was pointed out to the author by G. Ya. Popov to whom thanks are expressed here.

5. THE MIXED PROBLEM

The mixed problem consists of determining the functions [6]

$$\Phi_{1,2}(z) = \Phi(z) \pm i\Omega(z)/\sqrt{\varkappa} \tag{5.1}$$

with the boundary conditions

$$\Phi_m^+(t)+(-1)^m i\sqrt{\varkappa}\Phi_m^-(t)=2g_m(t), \quad t\in L, \quad m=1,2 \tag{5.2}$$
$$2g_{1,2}(t)=(\sigma_y-i\tau_{xy})^+\mp 2i\mu(u'+iv')^-/\sqrt{\varkappa}$$

The conditions for the displacements to be single-valued on passing around the cuts L_n in this case have the following form according to relations (1.1), (5.1):

$$\int_{L_n}[\Phi_1^+(t)+\Phi_2^+(t)]\,dt=\frac{2}{\varkappa+1}\int_{L_n}[(\sigma_y-i\tau_{xy})^+ + 2\mu(u'+iv')^-]\,dt \tag{5.3}$$

Let the principal vectors P_n of external forces applied to the edges of the cuts L_n also be specified. Then

$$\int_{L_n}[\Phi_1^+(t)-\Phi_2^+(t)]\,dt=\frac{2\sqrt{\varkappa}}{\varkappa+1}P_n+$$
$$+\frac{2i}{\varkappa+1}\int_{L_n}\left[(\sigma_y-i\tau_{xy})^+-\frac{2\mu}{\varkappa}(u'+iv')^-\right]dt \tag{5.4}$$

Adding and subtracting conditions (5.3) and (5.4) we obtain

$$\int_{L_n}\Phi_1^+(t)\,dt=E_{n1}, \quad \int_{L_n}\Phi_2^+(t)\,dt=E_{n2}, \quad n=0,\pm 1,\ldots \tag{5.5}$$

where E_{n1} is the half-sum of the right-hand sides of Eqs (5.3) and (5.4) and E_{n2} is their half-difference.

We will take the solutions of problems (5.2) in the form

$$\Phi_m(z)=F_m(z)+X_m(z)Q_m(z)$$
$$F_m(z)=\sum_{k=-\infty}^{\infty}\frac{X_m(z)}{z-kT}\frac{1}{\pi i}\int_{L_k}\frac{\tau-kT}{X_m^+(\tau)}\frac{g_m(\tau)}{\tau-z}\,d\tau \tag{5.6}$$

$$X_m(z)=\left(\sin\frac{\pi(z+a)}{T}\right)^{-\gamma_m}\left(\sin\frac{\pi(z-a)}{T}\right)^{\gamma_m-1}\sin\frac{\pi z}{T}$$
$$\gamma_1=\frac{1}{4}-i\beta, \quad \gamma_2=\frac{3}{4}-i\beta, \quad \beta=\frac{\ln\varkappa}{4\pi}$$
$$Q_m(z)=A_m+A_{m0}z^{-1}+\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}A_{mk}\left(\frac{1}{z-kT}+\frac{1}{kT}\right) \tag{5.7}$$

where series (5.6) converges absolutely and uniformly in any unbounded region not containing the points of the line L, and the constants A_{mk} must be taken so that series (5.7) converges. Then, substituting the values of $\Phi_m^+(t)$ into conditions (5.5) we obtain, for each specific $m=1, 2$, the following system for determining the constants A_m, A_{mk}:

$$\delta A_m+\sum_{k=-\infty}^{\infty}\delta_{nk}A_{mk}=E_{nm}-\int_{L_n}F_m^+(t)\,dt, \quad n=0,\pm 1,\ldots$$
$$\delta=\int_{-a}^{a}X(t)\,dt, \quad \delta_{n0}=\int_{-a}^{a}\frac{X(t)\,dt}{t+nT}, \quad \delta_{nk}=\int_{-a}^{a}\frac{X(t)\,dt}{t+(n-k)T}+\frac{\delta}{kT}, \quad k\neq 0 \tag{5.8}$$

where $X(t)=X_m^+(t)$. Thus the problem of constructing the functions $\Phi_m(z)$ according to conditions (5.2) and (5.5) has been reduced to determining the solutions of system (5.8), which can be solved in the same way as system (2.11).

A similar system is obtained in the case when the differences of the displacements of the points $nT+T-a$ and $nT+a$ are specified instead of the numbers P_n.

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